A Primer on Dynamic Screening Contracts

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August 2, 2015

1 Static Screening Contract

A monopolist (principal) wants to sell a good to an agent (consumer) The agent's valuation of the good is his private information. The principal has a convex cost function. Preferences over price-quantity paris (p, q) are quasi-linear,

Principal: $p - \frac{q^2}{2}$
Agent: $\theta q - p$'

Note that the first best is given by the quantity that maximizes $\left(p - \frac{q^2}{2}\right) + (\theta q - p)$, that is:

$$q^{eff} = \theta.$$

The agent's private information can take two values $\Theta = \{\theta_H, \theta_L\}$, drawn from from a prior $\{\mu_H, \mu_L\}$. Let $\theta_H - \theta_L = \Delta\theta > 0$. The principal offers a take it or leave it contract that maximizes her expected utility subject to truthtelling and voluntary participation, that is, incentive compatibility and individual rationality, respectively.

$$\max_{(p_H,q_H),(p_L,q_L)} \quad \mu_H\left(p_H - \frac{q_H^2}{2}\right) + \mu_L\left(p_L - \frac{q_L^2}{2}\right)$$

subject to

$$\begin{aligned}
IC_H: &\quad \theta_H q_H - p_H \geq \theta_H q_L - p_L \\
IR_H: &\quad \theta_H q_H - p_H \geq 0 \\
IC_L: &\quad \theta_L q_L - p_L \geq \theta_L q_H - p_H \\
IR_L: &\quad \theta_L q_L - p_L \geq 0
\end{aligned}$$

A standard tool to "solve" this optimization problem is to consider a relaxed problem where we maximize the objective subject to IC_H and IR_L only.

$$(RP) \quad \max_{(p_H,q_H),(p_L,q_L)} \quad \mu_H\left(p_H - \frac{q_H^2}{2}\right) + \mu_L\left(p_L - \frac{q_L^2}{2}\right)$$

subject to

$$\begin{aligned} IC_H: &\quad \theta_H q_H - p_H \geq \theta_H q_L - p_L \\ IR_L: &\quad \theta_L q_L - p_L \geq 0 \end{aligned}$$

Lemma 1. *In (RP), IC_H binds at the optimum.*

Lemma 2. *In (RP), IR_L binds at the optimum.*

We follow proof by contradiction for both lemmas above. Suppose it does not, and then change the contract in way that increase the value of the objective. And, thus conclude that these constraints must bind at the optimum.

Therefore, we have

$$p_L = \theta_L q_L$$

$$p_H = \theta_H q_H - \Delta\theta q_L$$

Substituting these into the objective function we get

$$(RP) \quad \max_{q_H, q_L} \quad \mu_H \left(\theta_H q_H - \Delta\theta q_L - \frac{q_H^2}{2} \right) + \mu_L \left(\theta_L q_L - \frac{q_L^2}{2} \right)$$

The optimum solution is given by

$$\begin{aligned} q_H^* &= \theta_H \\ q_L^* &= \theta_L - \frac{\mu_H}{\mu_L} \Delta\theta \\ p_H^* &= \theta_H q_H^* - \Delta\theta q_L^* \\ p_L^* &= \theta q_L^* \end{aligned}$$

Lemma 3. *The solution to (RP) satisfies IR_H and IC_L.*

Thus, $\{(p_H^*, q_H^*), (p_L^*, q_L^*)\}$ is the solution to the original problem. Two important are immediate. First, the quantity for the high type is efficient, and quantity for the low type is distorted downwards. And, second the optimal contract is separating and monotonic in types.

Before we conclude the section on the static model, it is useful to consider implications of incentive compatibility for the agent's utility. Given, a menu of contracts, let the agent's utility vector by given by

$$\begin{aligned} u_H &= \theta_H q_H - p_H \\ u_L &= \theta_L q_L - p_L \end{aligned}$$

Note first that the whole optimization problem can be stated in terms of (u, q) instead of

(p, q). Moreover, IC_H and IC_L can respectively be written as

$$IC_H: \quad u_H - u_L \geq \Delta\theta q_L$$
$$IC_L: \quad u_L - u_H \leq \Delta\theta q_L$$

Let $\Delta u = u_H - u_L$. Since IC_H binds at the optimum, we have

$$\frac{\Delta u}{\Delta \theta} = q_L \tag{1}$$

For the continuous type space model, that is when $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, we have from incentive compatibility

$$(\theta - \theta')q(\theta') \leq u(\theta) - u(\theta') \leq (\theta - \theta')q(\theta)$$

that is,

$$q(\theta') \leq \frac{u(\theta) - u(\theta')}{\theta - \theta'} \leq q(\theta)$$

Thus, as $\theta' \to \theta$, we have

$$\frac{du(\theta)}{d\theta} = q(\theta) \tag{2}$$

Equations (1) and (2) are referred to as the envelope formula.

2 Dynamic Screening

Consider the same model as before. In addition, the monopolist and the consumer meet in the market twice and each period the good is sold. First period valuation is known to the agent when the contract is agreed upon. The second period valuation, $f(\theta'|\theta)$, is drawn from the following stochastic matrix,

	H	L
H	α_H	$1 - \alpha_H$
L	$1 - \alpha_L$	α_L

The agent should be given a minimum expected utility each period. Thus, individual rationality constraint is imposed in each period. Checking for incentive compatibility can be very complicated in a dynamic setting for the agent can resort to complicated history dependent misreports. Luckily, a simple result drastically reduces the number of deviations we need to consider.

One deviation principle: Without loss of generality, at every history we can restrict attention to deviations in which the agent lies once and then tells the truth again in the future.

The revelation principle holds. The principal offers a (possibly) history dependent menu $\left(p_i, q_i, (p_i(h), q_i(h))_{h=H,L}\right)_{i=H,L}$ to maximize her expected profit subject to incentive compatibility and individually rational constraints for every history,

$$IR_H, IR_L, IC_H, IC_L,$$

$$IR_H(h), IR_L(h), IC_H(h), IC_L(h),$$

for $h = H, L$.

The expected utility of an agent of type i in the first period is given by

$$U_i = \theta_i q_i - p_i + \delta \sum_{j=H,L} f(\theta_j|\theta_i)(\theta_j q_j(i) - p_j(i)),$$

and in the second period after history $h = H, L$ by

$$u_i(h) = \theta_i q_i(h) - p(h)$$

Then, a sample of constraints are as follows

$$\begin{aligned} IC_H: \quad U_H &\geq \theta_H q_L - p_L + \delta[\alpha_H u_H(L) + (1-\alpha_H)u_L(L)] \\ &= U_L + \Delta q_L + \delta(\alpha_H - (1-\alpha_L))(u_H(L) - u_L(L)) \end{aligned}$$

$$IR_L: \quad U_L \geq 0$$

$$IC_H(h): \quad u_H(h) \geq u_L(h)\theta q_L$$

$$IR_L(h): \quad u_L(h) \geq 0$$

The principal profit is given by

$$\mu_H R_H + \mu_L R_L, \tag{$\star$}$$

where

$$\begin{aligned} R_i =& p_i - \frac{q_i^2}{2} + \delta \sum_{j=H,L} f(\theta_j|\theta_i)\left(p_j(i) - \frac{(q_j(i))^2}{2}\right) \\ =& \theta_i q_i - \frac{q_i^2}{2} + \delta \sum_{j=H,L} f(\theta_j|\theta_i)\left(\theta_j q_j(i) - \frac{(q_j(i))^2}{2}\right) - U_i \end{aligned}$$

Consider a relaxed problem of maximizing ($\star$) subject to only

$$IR_L, IC_H, IR_L(h), IC_H(h),$$

for $h = H, L$.

Lemma 4. *All the constraints in the relaxed problem can be assumed to hold as equalities.*

This gives

$$u_L(h) = \theta_L q_L(h) - p_L(h) = 0.$$

$$u_H(h) = u_L(h) + \Delta\theta q_L(h) = \Delta\theta q_L(h).$$

Thus,

$$\boxed{\frac{u_H(h) - u_L(h)}{\Delta\theta} = q_L(h)}.$$

Also,

$$U_L = 0$$

and,

$$\begin{aligned} U_H =& U_L + \Delta\theta q_L + \delta\left(\alpha_H - (1-\alpha_L)\right)\left(u_H(L) - u_L(L)\right) \\ =& \Delta\theta q_L + \delta\left(\alpha_H - (1-\alpha_L)\right)\Delta\theta q_L(L) \end{aligned}$$

Thus,

$$\boxed{\frac{U_H - U_L}{\Delta\theta} = q_L + \delta\left(\alpha_H - (1-\alpha_L)\right) q_L(L)}$$

Note that

$$\frac{U_H - U_L}{\Delta\theta} = q_L + \delta\left(\alpha_H - (1-\alpha_L)\right)\frac{\Delta u(L)}{\Delta\theta} \tag{3}$$

is the dynamic analogue of the envelope theorem in the static models. It states the marginal impact of misreporting on today and tomorrow's utility. Substituting U_H and U_L back in ($\star$), we get the objective function only in terms of quantities.

Proposition 1. *The solution to the relaxed problem is given by*

$$q_H = \theta_H \quad q_L = \theta_L - \frac{\mu_H}{\mu_L}\Delta$$

$$q_H(H) = \theta_H \quad q_L(H) = \theta_L$$

$$q_H(L) = \theta_H \quad q_L(L) = \theta_L - \frac{\mu_H}{\mu_L}\frac{\alpha_H - (1-\alpha_L)}{\alpha_L}\Delta$$

Properties. *Generalized no distortion at the top:* once the type becomes the high type- the contract becomes efficient.
Vanishing distortions at the bottom: At the lowest history with lowest types in each period the distortions decrease over time.

Special cases. iid and constant types.

For the continuous types model, we can similarly get the envelope theorem as

$$\frac{dU(\theta}{d\theta} = q(\theta) + \delta\int_{\Theta} q(\theta'|\theta)\frac{\partial F(\theta'|\theta)}{\partial\theta}d\theta'$$